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A NECESSARY CONDITION FOR THE CONTROLLABILITY OF A NON-LINEAR SYSTEM†

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A necessary condition for total controllability (in an arbitrary time) of a non-linear real analytical dynamical system is proved. In the linear case, it reduces to the well-known Kalman criterion [1]. Unlike the previously known non-linear generalizations of the Kalman condition [2, 3] the proposed condition follows from global considerations based on the construction of semi-permeable surfaces for the dynamical system under consideration.

1. THE KALMAN complete controllability criterion [1] is well known in the theory of linear controlled systems. In order for the system

$$x^{\cdot} = Ax + Bu, x \in V, u \in U \tag{1.1}$$

where $A: V \to V$, $A: V \to V$ are linear operators, to be completely controllable, it is necessary and sufficient that

$$\operatorname{rank}(B, AB, \ldots, A^{n-1}B) = n, \ n = \dim V \tag{1.2}$$

Some non-linear analogues of the Kalman criterion (1.2) are also known, relating to non-linear controlled systems of the form

$$x^{\cdot} = f(x) + g(x)u, \ x \in V, u \in U \tag{1.3}$$

Here it already makes sense to regard V as merely a smooth manifold and not necessarily a vector space, f as a smooth vector field on V, and q as a smooth linear map of the vector space U into the tangent bundle TV of the manifold V. A necessary condition for the complete controllability of system (1.3) in the real-analytic case consists [2, 3] in the fact the Lie algebra of the vector fields generated by the fields f and gu, for all possible $u \in U$, has the maximum point $x \in V$

$$\operatorname{rank}_x \operatorname{Lie}(f, gu) = \dim V \tag{1.4}$$

(this means that repeated commutators of the fields f and gu at each point $x \in V$ generate the tangent space T_xV). We know that for system (1.1) the above condition turns into the Kalman condition, and is therefore necessary and sufficient. In the general case, condition (1.4) is only necessary, as is clear from the following trivial example. Suppose

$$V = R^2 = Re_1 \neq Re_2, \ f = e_1, g = e_2, u \in R \tag{1.5}$$

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Condition (1.4) is of course satisfied, however under motion along the trajectories of system (1.3) the coordinate x_1 increases monotonically, which of course contradicts the total controllability.

In this paper, it is shown that condition (1.4) can be strengthened to

$$\operatorname{rank}_x I(gu) = \dim V \tag{1.6}$$

where $I(gu)$ denotes the ideal generated by the fields gu in the Lie algebra $\mathrm{Lie}(f, gu)$, while rank_x (or $\dim_x$) are the dimensions of the subspace in the tangent space at the point x. For system (1.1) condition (1.6) is again equivalent to the Kalman condition.

Note that example (1.5) already fails to satisfy condition (1.6) since $I(gu)$ in this situation is identical with Re_2. At the same time a minor complication of example (1.5) shows that condition (1.6) is after all again insufficient.

Indeed, suppose

$$\begin{gathered} f(x) = \varphi(x_2)e_1, \; g = e_2 \\ V = R^2 = \{(x_1, x_2) = x_1e_1 + x_2e_2\}, \; u \in R \end{gathered} \tag{1.7}$$

where $\varphi > 0$ and $\varphi' > 0$. Then $I(gu)$ contains the field $g = e_2u[g, f] = \varphi' e_1$, and hence condition (1.6) is satisfied, while the first coordinate x_1 of the phase vector increases monotonically with time, which contradicts controllability.

2. We consider the controlled dynamical system (1.3). We assume that V is a real analytic manifold and that the fields f and gu are real and analytic. We also assume that the first group of real homologies of the manifold V is null

$$H_1(V, R) = 0 \tag{2.1}$$

Condition (2.1) will be used in the following equivalent form. Suppose $p: M \to V$ is a covering whose deck transformation group $\Gamma = \pi_1(V)/p_*\pi_1(M)$ is a sub-group of the additive group of real numbers. Then p is an isomorphism.

Condition (2.1) is satisfied, for example, when $V = R^n$ or S^n (an n-dimensional sphere) and is not satisfied when $V = (S^1)^n$ (an n-dimensional torus).

Theorem. Condition (1.6) is necessary for the complete controllability (over an arbitrary time) of system (1.3).

Proof. Assume the contrary. Suppose that condition (1.6) is not satisfied at at least one point $x \in V$, while system (1.3) is completely controllable. From the complete controllability it follows that $\operatorname{rank}_x I(gu)$ does not depend in the point $x \in V$.

Indeed suppose $I \subset \mathrm{Lie}(X)$ is an arbitrary ideal in the Lie algebra generated by the vector fields X_u depending on the parameter $u \in U$, and I_x is the restriction of the fields of I to the tangent space at the point x.

Then, denoting by e^{tX} the phase flow corresponding to the field X, and by $(e^{tX})_*$ its action on vector fields, we find that if $I \subset g$ is an ideal in the Lie algebra g consisting of vector fields on V, $x \in g$, and IO_V is the sub-sheaf of the bundle of the sheaf of vector fields on G generated by the ideal I, then

$$(e^{tX})_* IO_V \subset IO_V \tag{2.2}$$

Assertion (2.2) follows from the finite generation of IO_V as a sheaf of modules over O_V, which in turn follows because the ring of germs of analytic functions is Noetherian.

From this it follows that if $y = e^{tX}x$, then

$$I_y = (e^{tX})_* I_x \tag{2.3}$$

and, in particular

$$\dim I_y = \dim I_x \tag{2.4}$$

(We remark in passing that condition (1.4) immediately follows from the fact that $\dim g_x$ does not depend on $x \in V$ (special case (2.4)) and the Frobenius' theorem).

By applying relation (2.4) repeatedly in the situation being considered and using the complete controllability condition we obtain

$$\text{rank}_y\, I(gu) = \text{rank}_x\, I(gu) \tag{2.5}$$

for any pair of points x, y.

The ideal $I(gu)$ has in the Lie algebra Lie(f, gu) a codimension not greater than one, because the element in f in $I(gu)$ generates Lie(f, gu) as a vector space. Then, using the necessary condition (1.4) for complete controllability, relation (2.5) and the assumption that condition (1.6) is not satisfied, we find that the ideal $I(gu)$ defines a foliation F on V of codimension 1. Furthermore, there is a field f, everywhere transverse to the foliation and such that its phase flux $\Phi_t = e^{tf}$ (by (2.2)) takes F into itself. (If the field F does not generate a one-parameter group, i.e. the corresponding differential; equation can only be solved for short times, then one can replace f by a field rf where $r > 0$ is a smooth (or real analytic) functions that decreases sufficiently rapidly at infinity, which itself generates a one-parameter group.)

We consider the map

$$\Phi: R \times L \to V$$
$$\Phi(t, x) = \Phi_t(x) = e^{tf}(x) \tag{2.6}$$

where L is some fixed leaf of our foliation F.

We claim that Φ is a covering and its deck transformation group is a sub-group of real numbers R.

Indeed, Φ is a local isomorphism by virtue if the transversatility of the field f to the foliation F and, if $U \subset R \times L$ is an open subset isomorphic to $\Phi(U)$, then

$$\Phi^{-1}(\Phi(U)) = \bigcup_{t \in \Gamma} \Phi_t(U) = U \times \Gamma,$$

where $\Phi_t(\tau, x) = (\tau - t, \Phi_t(x))$ and $\Gamma \subset R$ is the set of those values $t \in R$ for which $\Phi_t(L) = L$. The reason for this decomposition of $\Phi^{-1}(\Phi(U))$ is that if the intersection $\Phi_t(L) \cap L$ is non-empty, then $\Phi_t(L) = L$ because $\Phi_t(F) = F$. Obviously the deck transformation group Γ of the covering Φ is a sub-group of R. The conditions of the theorem (see (2.1)), however, prohibit non-trivial coverings of this type over the manifold V. It remains to conclude that Φ is an isomorphism. Hence there is a global coordinate t in V which increases montonically along the trajectories of system (1.3). Such a situation is of course inconsistent with complete controllability.

The theorem is proved.

Note that the only place in the proof which does not carry through in the infinitely differentiable case is relation (2.2). It is however, satisfied if the sheaf IO_V is finitely generated over O_V. (Here O_V is the sheaf of germs of infinitely smooth functions.)

We will conclude with an example showing that one cannot totally avoid condition (2.1). To this end we modify example (1.5) as follows. We take $V = T^2 = R^2/Z^2$ to be the two-dimensional torus, $f = \alpha e_1 + e_2$, $g = e_2$, $u \in R$. Here e_1 and e_2 are unit coordinate vectors in $R^2 = T_x V$ and α is an irrational constant. It is obvious that $\text{rank}_x I(gu)_x = 1$ and condition (1.6) is not satisfied. Nevertheless in this case, system (1.3) is completely controllable. which follows from the density of the integral curve of the field f in V. (For more general results of this kind see [6].)

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